# Existence of Entire Solutions for Superlinear Elliptic Problems in $R^N$(Nonlinear Analysis and Convex Analysis)



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# Existence of Entire Solutions for Superlinear Elliptic Problems in $R^N$

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**1. Introduction .** In this talk, we are concerned with positive solutions of the following problem:

$$(P) \qquad \begin{cases} -\Delta u + u = g(x,u), & u > 0, \quad \text{in } R^N \\ u \in H^1(R^N), & N \geq 2 \end{cases}$$

where $f : R^N \to R$ and $g : \Omega \times R \to R$ is continuous with $g(x,0) = 0$ for $x \in \Omega$. In the last decade, the existence and the properties of the solutions of problem (P) has been studied by many authors. Recently, the existence of positive solutions of semilinear elliptic problem

$$(P_Q) \qquad \begin{cases} -\Delta u + u = Q(x) \mid u \mid^{p-1} u, & x \in R^N \\ u \in H^1(R^N), & N \geq 2 \end{cases}$$

has been studied by several authors, where $1 < p$ for $N = 2$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3$, $Q(x)$ is positive bounded continuous function. If the function $Q(x)$ is a radial function, the existence of infinity many solutions of problem $(P_Q)$ can be shown by restricting our attention to the radial functions(cf. [1]). In case that $Q(x)$ is nonradial, we encounter a difficultly caused by lack of compact embedding of Sobolev type. In [6,7], P.L. Lions presented a method, called concentrate compactness method, which enable us to solve problems with lack of compactness, and established the following result: Assume that

$$lim_{|x|\to\infty} Q(x) = \overline{Q}(>0) \quad \text{and} \quad Q(x) \geq \overline{Q} \text{ on } R^N,$$

then problem $(P_Q)$ has a positive solution. This result is based on the observation that the ground state level $c_Q$ of the functional

$$I_Q(u) = \frac{1}{2}\int_{R^N}(|\nabla u|^2 + |u|^2)dx - \frac{1}{p+1}\int_{R^N} Q(x)u^{p+1}dx$$

is lower than the ground state level $c_{\overline{Q}}$ of functional $I_{\overline{Q}}$. We can apply the concentrate compactness method problem (P) to the problem in case that $g : R^N \times R \to R$ satisfies $\lim_{|x|\to\infty} g(x,t) = t^p$ and the least critical level $c_1$ of the functional

$$I(u) = \frac{1}{2}\int_{R^N}(|\nabla u|^2 + |u|^2)dx - \int_{R^N}\int_0^{u(x)} g(x,t)dtdx,$$

$u \in H^1(R^N)$, is lower than that of

$$I^\infty(u) = \frac{1}{2}\int_{R^N}(|\nabla u|^2 + |u|^2)dx - \frac{1}{p+1}\int u^{p+1}dx.$$

Under additional conditions on $g$, the exsitence of positive solutions (P) was established by Ding & Ni[4] and Stuart[10]. Recently, Cao[2] proved the existence of positive solution of $(P_Q)$ for the case that $c_Q \le c_{\overline{Q}}$ under the hypothesis that $\lim_{\|x\|\to\infty} Q(x) = \overline{Q}$ and $Q(x) \ge 2^{(1-p)/2}\overline{Q}$ on $R^N$. In case that $c_Q = c_{\overline{Q}}$, we encounter a difficulity, bacause we can not apply the concentrate compactness method directly. On the other hand, in case that $g$ is not given by the form $Q(x)t^p$, we have to overcome another difficulity: that is, we can not use the Lagrange's method of indeterminate coefficients. In the problem $(P_Q)$, we find a solution $u$ of minimizing problem

$$\inf\{I_Q(u) : u \in V_\lambda\},$$

$$V_\lambda = \{u \in H^1(R^N), u > 0, \int_{R^N} Q(x)u^{p+1}dx = 1\}$$

Then $cu$ is a solution of $(P_Q)$ for some $c > 0$. The Lagrange's method does not work if $g$ is not the form $Q(x)t^p$. Our approach enable us to treat the problem (P) with $g$ satisfying that $g(0) = 0$ and $g(t) \to t^p$ as $t \to \infty$. We also consider the nonhomoginous case:

$$(P_f) \qquad \begin{cases} -\Delta u + u = |u|^{p-1}u + f, & x \in R^N \\ u \in H^1(R^N), & N \ge 3 \end{cases}$$

where $p > 1$ for $N = 1$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3$.

The nonhomogeneous problem $(P_f)$ was studied by Zhu[12]. In [12], the existence of at least two solutions of (P) was proved for nonnegative functions $f \in L^2(R^N)$ with a small $L^2$−norm and a exponential decay

$$f(x) \leq Cexp\{-(1+\epsilon) \mid x \mid\}, \qquad \text{for } x \in R^N.$$

In the present paper, we consider multiple existence of solutions of (P) for nonnegative functions $f \in L^q(R^N)$, where $q = (p+1)/p$. Our result does not require that $f \in L^\infty(R^N)$ or any condition for the decay of $f$ at infinity.

In this talk, we show an approach for problems $(P)$ and $(P_f)$ based on arguments using singular homology theory. Throughout this paper, we denote by $| \cdot |_q$ the norm of $L^q(R^N)$. We impose the following conditions on the continuous mapping $g : R^N \times R \to R$:

(g1) There exists a positive number $d < 1$ such that

$$-dt + (1-d)t^p \leq g(x,t) \leq dt + (1+d)t^p$$
$$\text{for all } (x,t) \in R^N \times [0,\infty);$$

(g2) there exists a positive number $C$ such that

$$| g_t(x,0) | < 1 \quad \text{and} \quad 0 < t^2 g_{tt}(x,t) < C(1+t^p)$$
$$\text{for all } (x,t) \in R^N \times [0,\infty);$$

(g3)
$$\lim_{|x| \to \infty} g(x,t) = | t |^{p-1} t$$
$$\text{uniformly on bounded intervals in } [0,\infty),$$

where $1 < p$ for $N = 2$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3$, and $g_t(\cdot,\cdot)$ stands for the derivative of $g$ with respect to the second variable.

We can now state our main results.

**Theorem 1.** *Suppose that (g2) and (g3) holds. Then there exists $d_0 > 0$ such that if (g1) holds with $d < d_0$ , then problem (P) has a positive solution.*

For problem $(P_f)$, we have

**Theorem 2.** *There exists a positive number $C$ such that for each $f \in L^q(R^N)$, with $f \geq 0$ and $| f |_q < C$, problem $(P_f)$ possesses at least two solutions.*

**2. Preliminaries.** We just give a sketch of a proof of Theorem 1 to show that how the singular homology theory works for the proof of existence of positive solutions. We put $H = H^1(R^N)$. Then $H$ is a Hilbert space with norm

$$\| u \| = \left(\int_{R^N} (| \nabla u |^2 + | u |^2)dx\right)^{1/2}.$$

The norm of the dual space $H^{-1}(R^N)$ of $H$ is also denoted by $\| \cdot \|$. $B_r$ stands for the open ball centered at 0 with radius $r$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $H^1(R^N)$ and $H^{-1}(R^N)$. For each $r > 1$, the norm of $L^r(R^N)$ is denoted by $| \cdot |_r$. For simplicity, we write $| \cdot |_*$ instead of $| \cdot |_{p+1}$. For $u \in H$, we set $u^+(x) = \max\{u(x), 0\}$. We denote by $C_p$ the minimal constant satisfying

$$| u |_* \leq C_p \| u \| \qquad \text{for } u \in H. \tag{2.1}$$

It is easy to check that critical points of $I$ are solutions of (P). It is also obvious that nonzero critical points of $I^\infty$ are solutions of (P) with $g(t) = t^p$ for $t \geq 0$. For each functional $F$ on $H$ and $a \in R$, we set $F_a = \{u \in H : F(u) \leq a\}$. We put

$$M = \{u \in H \backslash \{0\} : \| u \|^2 = \int_{R^N} ug(x,u)dx\}$$
$$M^\infty = \{u \in H \backslash \{0\} : \| u \|^2 = \int_{R^N} u^{p+1}dx\}$$

For the proof of the following two propositions are crucial:

**Proposition 2.1.** *There exists positive number $d_0 < \widetilde{d}_0$ and $\epsilon_0$ satisfying that if (g1) holds with $d \leq d_0$, then for each $0 < \epsilon < \epsilon_0$,*

$$H_*(I^\infty_{c+\epsilon}, I^\infty_\epsilon) = H_*(I_{c+\epsilon}, I_\epsilon)$$

*where $H_*(A, B)$ denotes the singular homology group for a pair $(A, B)$ of topological spaces(cf. Spanier[8]).*

**Proposition 2.2.** *For each positive number $\epsilon < \epsilon_0$,*

$$H_q(I^\infty_{c+\epsilon}, I^\infty_\epsilon) = \begin{cases} 2 & \textit{if } q = 0, \\ 0 & \textit{if } q \neq 0. \end{cases}$$

Here we give a proof for Proposition 2.2.

We set

$$T_{u_\infty}(M^\infty) = \{\lim_{t\to 0}(c(t)-u_\infty)/t : c \in C^1((-1,1);M^\infty) \text{ with } c(0)=u_\infty\},$$

$$\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+ = \{-\tau_x u_\infty : x \in R^N\} \cup \{\tau_x u_\infty : x \in R^N\}$$

and

$$T_{u_\infty}(\mathcal{C}) = \{\lim_{t\to 0}(u_\infty(\cdot + tx) - u_\infty(\cdot))/t : x \in R^N\}.$$

It follows from the definition of $M^\infty$ that the codimension of $T_{u_\infty}(M^\infty)$ in $H$ is one. It is also obvious that $\dim T_{u_\infty}(\mathcal{C}) = N$. We denote by $\widetilde{H}$ the subspace such that $H = \widetilde{H} \oplus T_{u_\infty}(\mathcal{C})$. For each $r > 0$, we set $B_r^0 = B_r \cap \widetilde{H}$. Here we consider the linealized equation

$$(L) \qquad -\Delta u + u - h(x)u = \mu u, \qquad u \in H, \mu \in R,$$

where $h(x) = p \mid u_\infty(x) \mid^{p-1}$ for $x \in R^N$. Since $-\Delta$ is positive definite and $h(x)I$ is compact, we find by Freidrich's theory that the negative spectrums of $A = -\Delta - h(x)I$ are finite and each eigenspace corresponding to a negative eigenvalue is finite dimensional. Then each eigenspace corresponding to a nonpositive eigenvalue of $L = -\Delta + I - h(x)I$ is finite dimensional. Then there exists $c_0 > 0$ and a decomposition $H = H_- \oplus H_0 \oplus H_+$ such that $H_0 = ker(L)$ and $L$ is positive(negative) definite on $H_+(H_-)$ with

$$\langle Lv, v\rangle \geq c_0 \parallel v \parallel^2 \ (\leq -c_0 \parallel v \parallel^2) \qquad \text{for } v \in H_+(H_-).$$

Since each $u \in \mathcal{C}$ is a solution of problem $(P_\infty)$, we can see that $T_{u_\infty}(\mathcal{C}) \subset H_0$.

**Lemma 2.3.** *$\dim H_- = 1$.*

**Proof.** Since $I^\infty$ attains its minimal on $M^\infty$ at $u_\infty$, we have that $T_{u_\infty}(M^\infty) \subset H_+ \oplus H_0$. Then since the codimension of $M^\infty$ is one, we find that $\dim H_- \leq 1$. On the other hand, we have

$$\begin{aligned}\langle Lu_\infty, u_\infty\rangle &= \int_{R^N} (\mid \nabla u_\infty \mid^2 + \mid u_\infty \mid^2 - p \mid u_\infty \mid^{p+1}) dx \\ &< \int_{R^N} (\mid \nabla u_\infty \mid^2 + \mid u_\infty \mid^2 - \mid u_\infty \mid^{p+1}) dx = 0.\end{aligned} \tag{2.2}$$

Then we have that $\dim H_- \geq 1$. This completes the proof. ∎

In the following we denote by $\varphi$ an element of $H_-$ with $\| \varphi \| = 1$. Here we note that since $h \in C^\infty(R^N)$, each solution $u$ of (L) is in $C^1(R^N)$. It then follows that if $u$ has the form

$$u(r, \theta) = \psi(r)\xi(\theta_1, \cdots, \theta_{n-1}), \qquad \text{with } \xi \not\equiv \text{ const.},$$

in spherical coordinate, $\psi$ satisfies that $\psi(0) = 0$.

We denote by $H_r$ the set of all radial functions in $H$ and by $(L_r)$ the problem $(L)$ restricted to $H_r$. Then, in spherical coordinates, the problem $(L_r)$ with $\mu > 0$ is reduced to

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + (h-1)\psi = -\mu\psi(r), \qquad r > 0, \psi \in C_r, \quad (2.3)$$

$$\frac{d\psi(r)}{dr}(0) = 0, \quad (2.4)$$

where $C_r = \{\psi \in C[0, \infty) : \lim_{r\to\infty} \psi(r) = 0\}$.

We next consider nonradial solutions of (L). In case of nonradial functions, the problem (L) is deduced to

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h-1) - \frac{\alpha_k}{r^2})\psi(r) = -\mu\psi(r), \qquad r > 0, \psi \in C_r (2.5)$$

$$\psi(0) = 0 (2.6)$$

where $\alpha_k = k(k+n-1)$, $k = 1, 2, \cdots$. Note that $\alpha_k$ are the eigenvalues of Laplacian $-\Delta$ on $S^{n-1}$, the unit sphere, and the dimension of the eigenspace $S_k$ associate with $\alpha_k$ is

$$\rho_k = \binom{k+n-2}{k} \frac{n+2k-2}{n+k-2}.$$

That is there exists smooth functions $\{\varphi_{k,i} : i = 1, \cdots, \rho_k\}$ defined on $S^{n-1}$ such that $S_k = span\{\varphi_{k,1}, \cdots, \varphi_{k,\rho_k}\}$, and the functions $u = \psi(r)\varphi_{k,i}(\theta)$ are the solutions of (L).

**Lemma 2.4.** *$\dim H_0 \leq N + 1$.*

**Proof.** Since $\dim H_- = 1$ and $u_\infty \in H_r$, we have by (2.2) that the problems (2.3) , (2.4) has exactly one negative eigenvalue. We also note that each nonpositive eigenvalue $\mu$ of problems (2.3) , (2.4) is simple. Then the dimension of $H_{0,r} = H_0 \cap H_r$ is at most one.

We next consider nonradial cases. That is we will see that the eigenspace of the problem (2.5) with $\mu = 0$ is N-dimensional space. Recalling that $\nabla I(v) = 0$ on $\mathcal{C}$, we can see that

$$-\Delta v + v - h(x)v = 0 \qquad \text{for all } v \in T_{u_\infty}(\mathcal{C}). \tag{2.7}$$

That is $T_{u_\infty}(C) \subset H_0$. Since $\dim T_{u_\infty}(C) = N$, we have that $\dim H_0 \geq N$. On the other hand, since $u_\infty$ satisfies

$$u''(r) + \frac{n-1}{r}u'(r) + p \mid u_\infty \mid^{p-1} u(r) = 0, \tag{2.8}$$

we find that $v(r) = u'_\infty$ satisfies

$$v''(r) + \frac{n-1}{r}v'(r) + ((h(x) - 1) - \frac{\alpha_1}{r^2})v(r) = 0.$$

Then we find that the N-dimensional space $\widetilde{C} = span\{v(r)\varphi_{1,i} : i = 1, \cdots, n-1\}$ is a subspace of solution set of (L) with $\mu = 0$. We claim that there exists no nonradial solution of (L) with $\mu = 0$ which is not contained in $\widetilde{C}$. Suppose contrary, there exists a nonradial solution $z$ of (L) with $\mu = 0$ such that $z \perp \widetilde{C}$. Then there exists $\psi \in C_r$ such that

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h(x) - 1) - \frac{\alpha_k}{r^2})\psi(r) = 0$$

for some $k > 1$ and $z = \psi(r)\varphi_{k,i}$ are solutions of (L) with $\mu = 0$. The equality above can be rewritten as

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h(x) - 1) - \frac{(\alpha_k - \alpha_1)}{r^2})\psi(r) - \frac{\alpha_1}{r^2}\psi(r) = 0.$$

Then $u = \psi(r)\varphi_{1,1}$ is a soluiton of problem

$$-\Delta u + u - h(x)u = \frac{(\alpha_1 - \alpha_k)}{r^2}u.$$

It then follows that

$$< -\Delta u + u - h(x)u, u >< 0. \tag{2.9}$$

Since $u$ is orthogonal to $\varphi$, we obtain from (2.9) that $\dim H_- \geq 2$. This is a contradiciton. Thus we obtain that $H_0 = T_{u_0}(\mathcal{C}) \oplus H_{0,r}$ and then $\dim H_0 \leq N+1$. ∎

Here we recall that $H$ has a decomposition $H = \widetilde{H} \oplus T_{u_\infty}(\mathcal{C})$ and then $H = \tau_x \widetilde{H} \oplus \tau_x T_{u_\infty}(\mathcal{C})$ for each $x \in R^N$. Then since $\mathcal{C}_\pm$ are smooth $N-$manifolds, we have that there exists $r_0 > 0$ such that

$$\tau_x((-1)^i u_\infty + B^0_{r_0}) \cap \tau_y(u_\infty + B^0_{r_0}) = \phi \tag{2.10}$$

for all $x, y \in R^N$ with $x \neq y$, and $i = 0, 1$. Here we consider a restriction $I^\infty \mid_{u_\infty + \widetilde{H}}$ of $I^\infty$ on $u_\infty + \widetilde{H}$. Then from Lemma 3.2 and Lemma 3.3, we have by Gromoll-Meyer theory[3] that there exists subspaces $H_1$ $H_{2,1}$, $H_{2,2}$ of $\widetilde{H}$, a positive number $r_1 < r_0$, a mapping $\beta \in C^1((H_{2,2} \cap B^0_{r_1}), R)$ and a homeomorphism $\psi : u_\infty + B^0_{r_1} \to u_\infty + \widetilde{H}$ such that $\widetilde{H} = H_1 \oplus H_{2,1} \oplus H_{2,2}$ and

$$I^\infty \mid_{u_\infty + \widetilde{H}} (\psi(u)) = c - \| u_1 \|^2 + \| u_{2,1} \|^2 + \beta(u_{2,2}) \tag{2.11}$$

for each $u \in u_\infty + B^0_{r_1}$ with $u = u_\infty + u_1 + u_{2,1} + u_{2,2}$, $u_1 \in H_1$, $u_{2,i} \in H_{2,i}$, $i = 1, 2$. It follows from Lemma 2.3 that $H_{2,2}$ is one dimensional. Noting that $T_{u_\infty}(M) \subset H_0 \oplus H_+$ and $u_\infty$ is the minimal point of $I^\infty$ on $M$, we have by choosing $r_1$ sufficiently small that $\beta(t\varphi_2)$ is strictly increasing as $\mid t \mid$ increases in $[-r_1, r_1]$, where $\varphi_2 \in H_{2,2}$ with $\| \varphi_2 \| = 1$.

Since $I^\infty$ is even, it is obvious that $I^\infty$ has the form (2.11) on $-(u_\infty + B^0_{r_1})$. We also note that for each $x \in R^N$, (2.11) holds for each $u \in \tau_x(u_\infty + B^0_{r_0})$ with $\psi$ replaced by $\tau_{-x} \circ \psi$.

**Proof of Proposition 2.2.** By the deformation property(cf. theorem 1.2 of Chang[3]) and the homotopy invariance of the homology groups, we have

$$H_q(I^\infty_{c+\epsilon}, I^\infty_{c-\epsilon}) \cong H_q(I^\infty_c, I^\infty_{c-\epsilon}), \text{ and}$$
$$H_q(I^\infty_c \backslash \mathcal{C}, I^\infty_{c-\epsilon}) \cong H_q(I^\infty_{c-\epsilon}, I^\infty_{c-\epsilon}) \cong 0.$$

From the exactness of the singular homology groups ,

$$H_q(I^\infty_c \backslash \mathcal{C}, I_{c-\epsilon}) \to H_q(I^\infty_c, I^\infty_{c-\epsilon}) \to H_q(I^\infty_c, I^\infty_c \backslash \mathcal{C})$$
$$\to H_{q-1}(I^\infty_c \backslash \mathcal{C}, I^\infty_{c-\epsilon}) \to \cdots$$

we find

$$0 \to H_q(I_c^\infty, I_{c-\epsilon}^\infty) \to H_q(I_c^\infty, I_c^\infty \backslash \mathcal{C}) \to 0.$$

That is

$$H_q(I_c^\infty, I_{c-\epsilon}^\infty) \cong H_q(I_c^\infty, I_c^\infty \backslash \mathcal{C}).$$

Noting that $\cup\{\tau_x(\pm u_\infty + B_{r_1}^0) : x \in R^N\}$ are disjoint open neighborhoods of $\mathcal{C}_\pm$ respectively, and that $I^\infty$ is invariant under the translations $\tau_x$, we find from the excision property and (2.11) that

$$\begin{aligned}
&H_*(I_{c+\epsilon}^\infty, I_\epsilon^\infty) \\
&\cong H_*(I_c^\infty, I_c^\infty \backslash \mathcal{C}) \\
&\cong H_*(I_c^\infty \cap (\cup_{i=\pm1} \cup_x \tau_x(iu_\infty + B_{r_1}^0)), \\
&\qquad\qquad\qquad I_c^\infty \cap (\cup_{i=\pm1} \cup_x \tau_x(iu_\infty + B_{r_1}^0) \backslash \mathcal{C})) \\
&\cong H_*(u_\infty + B_{r_1}^1, (u_\infty + B_{r_1}^1) \backslash \{u_\infty\}) \\
&\qquad\qquad \oplus H_*(-u_\infty + B_{r_1}^1, (-u_\infty + B_{r_1}^1) \backslash \{u_\infty\}) \\
&\cong H_*([0,1], \{0,1\}) \oplus H_*([0,1], \{0,1\}).
\end{aligned}$$

This completes the proof. ∎

**3. Proof of Theorem 1.** We next consider a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying the following conditions:

(1) $U \cap (-U) = \phi$;

(2) $\{\tau_x u_\infty : |\, x\, | \geq r\} \subset intK$ for some $r > 0$;

(3) $cl(I_{c+\epsilon} \cap K) \subset int(I_{c+\epsilon} \cap U)$;

(4) $H_{N-1}(I_{c+\epsilon} \cap U) = 1, \quad H_1(I_{c+\epsilon} \cap U) = 0$;

(5) $I_\epsilon$ is a strong deformation retract of $I_{c+\epsilon} \backslash (K \cup (-K))$;

(6) $H_{N-1}((I_{c+\epsilon} \cap U) \backslash K) = 2$ or $H_0((I_{c+\epsilon} \cap U) \backslash K) \geq 2$ *holds*.

**Proposition 3.1.** *There exists a triple $(U, K, \epsilon) \subset H \times H \times R^+$ which satisfies (1) - (6).*

We omit the proof of Proposition 3.1.

**Lemma 3.2.** *Suppose that there exist a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying (1)-(6). Suppose in addition that $H_{N-1}((I_{c+\epsilon} \cap U)\backslash K) \geq 2$. Then $H_N(I_{c+\epsilon}, I_\epsilon) \geq 2$.*

**Proof.** We put $\widetilde{K} = K \cup (-K)$. Since $I_\epsilon$ is a strong deformation retract of $I_{c+\epsilon}\backslash\widetilde{K}$, we find that

$$H_q(I_{c+\epsilon}\backslash\widetilde{K}, I_\epsilon) \cong H_q(I_\epsilon, I_\epsilon) \cong 0.$$

Then we have from the exactness of the singular homology groups of the triple $(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}, I_\epsilon)$ that

$$0 \to H_q(I_{c+\epsilon}, I_\epsilon) \to H_q(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}) \to 0.$$

That is

$$H_q(I_{c+\epsilon}, I_\epsilon) \cong H_q(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}).$$

From (1), we find

$$H_q(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}) \cong H_q(W, W\backslash K) \oplus H_q(-W, (-W)\backslash(-K))$$

where $W = I_{c+\epsilon} \cap U$. Then since $H_{N-1}(W\backslash K) \geq 2$, we have from (4) and the exactness of the sequence

$$\to H_q(W, W\backslash K) \to H_{q-1}(W\backslash K) \to H_{q-1}(W) \to H_{q-1}(W, W\backslash K) \to \tag{3.1}$$

with $q = N$ that $H_N(I_{c+\epsilon}, I_\epsilon) \cong H_N(W, W\backslash K) \oplus H_N(W, W\backslash K) \geq 2$. ∎

**Lemma 3.3.** *Suppose that $(U, K, \epsilon) \subset H \times H \times R^+$ satisfies (1) - (6). Suppose in addition that $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U)\backslash K) = 1$. Then $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ or $H_0(I_{c+\epsilon}, I_\epsilon) = 2$ holds.*

**Proof.** From the argument in the proof of Proposition 3.2, we have that $H_1(I_{c+\epsilon}, I_\epsilon) \cong H_1(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U)\backslash K) \oplus H_N(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U)\backslash K)$. Then since $H_1(I_{c+\epsilon} \cap U) = 0$, and $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U)\backslash K) = 1$, the assertion follows from the exactness of the sequence (3.1) with $q = 1$. ∎

We can now prove Theorem 1.

**Proof of Theorem.** Let $(U, K, \epsilon)$ be the triple constructed above. We have by Proposition 2.1 and Proposition 2.2 that $H_1(I_{c+\epsilon}, I_\epsilon) = 2$ and $H_q(I_{c+\epsilon}, I_\epsilon) = 0$ for $q \neq 1$. Now suppose that $(I_{c+\epsilon} \cap U)\backslash K$ is disconnected. Then since $H_0((I_{c+\epsilon} \cap U)\backslash K) \geq 2$, we find by Lemma 3.2 that $H_N(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. On the other hand, if $U\backslash K$ is connected, then $H_0(U\backslash K) = 1$. Then by Lemma 3.3, we have $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ or $H_0(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. Thus we obtain that there exists a positive solution of (P). ∎

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